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A New Subclass of Harmonic Univalent Functions Defined by a Linear Operator

Metkari, A. N. *[1], Sangle, N. D.[2], and Darus, M.[3]

[1]*Research Scholar, Department of Mathematics, Visvesvaraya Technological University, Belagavi, Karnataka-590018, India*
[2]*Department of Mathematics, D.Y. Patil College of Engineering and Technology, Kasaba Bawada, Kolhapur, Maharashtra-416006, India*
[3]*Department of Mathematical Sciences, Faculty of Science and Technology, Universiti Kebangsaan Malaysia, Malaysia*

E-mail: anand.metkari@gmail.com
*Corresponding author

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Abstract

In this paper, a new subclass of complex-valued harmonic univalent functions $f(z) = h(z) + \overline{g(z)}$ in the open disk $\mathcal{U} = \{z : z \in C \ and \ |z| < 1\}$ defined by using a linear operator. We investigate coefficient condition, extreme points, distortion and convex combination for this subclass.

Keywords: harmonic; univalent; salagean operator.

1 Introduction

Let $\mathcal{H}$ represent the family of continuous complex valued harmonic functions which are harmonic in the open unit disk $\mathcal{U} = \{z : |z| < 1\}$ and let $\mathcal{A} \subset \mathcal{H}$ containing analytic functions in $\mathcal{U}$. A harmonic function in $\mathcal{U}$ of the form $f = h + \overline{g}$, where $h \in \mathcal{A}$ also $g \in \mathcal{A}$. Here h is called analytic part and g is called co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and sense-preserving in $\mathcal{U}$ is that $\left|h^{'}(z)\right| > \left|g^{'}(z)\right|$(see [7]). To this end, without loss of generality, we may write

$$h(z) = z + \sum_{t=2}^{\infty} a_t z^t, \quad g(z) = \sum_{t=1}^{\infty} b_t z^t, \quad z \in \mathcal{U}. \tag{1}$$

Let the family of sense-preserving, harmonic and univalent functions $f(z) = h(z) + \overline{g(z)}$ denoted by $\mathcal{S_H}$ in $\mathcal{U}$ satisfying the condition $f_z(0) - 1 = f(0) = 0$. One simply shows that the sense-preserving property implies that $|b_1| < 1$. The ${\mathcal{S_H}}^0 \subset \mathcal{S_H}$ containing all functions of $\mathcal{S_H}$ such that $f_{\bar{z}}(0) = 0$, for detail see [7].

The geometric subclass and some coefficient bounds of the class $\mathcal{S_H}$ studied in [7]. The following standard introductory text book [9] can be referred for more basic results, also see ([1], [11]). In many articles different researchers found out several interesting results [8], [15], [16], [20], [21] and [22]. Related class and its subclassess are also studied by [4], [5], [6], [12], [13] and [14].

Motivated by earlier work of [3] and [6]. We investigate coefficient condition, convex combination, distortion and extreme points.

For $f \in \mathcal{S}$, the differential operator D^n $(n \in \mathcal{N}_0 = \mathcal{N} \cup \{0\})$ of f was introduced in [14]. For $f(z) = h(z) + \overline{g(z)}$ defined by (1), in [12] defined the modified Salagean operator of f as

$$D^n f(z) = D^n h(z) + (-1)^n \, \overline{D^n g(z)}$$

where

$$D^n h(z) = z + \sum_{t=2}^{\infty} t^n a_t z^t \qquad and \qquad D^n g(z) = \sum_{t=1}^{\infty} t^n b_t z^t.$$

Next, for functions $f = h + \overline{g} \in \mathcal{A}$ given by (1), in [6] defined multiplier transformations, we denote the modified multiplier transformation of f as

$$I^0_{\xi,\sigma} f(z) = D^0 f(z) = h(z) + \overline{g(z)},$$

$$I^1_{\xi,\sigma} f(z) = \frac{\xi D^0 f(z) + \sigma D^1 f(z)}{\xi + \sigma} = \frac{\xi \left(h(z) + \overline{g(z)}\right) + \sigma \left(zh^{'}(z) - \overline{zg^{'}(z)}\right)}{\xi + \sigma}, \tag{2}$$

$$I^n_{\xi,\sigma} f(z) = I^1_{\xi,\sigma} \left(I^{n-1}_{\xi,\sigma} f(z)\right), \quad (n \in \mathbb{N}_0). \tag{3}$$

For $0 \leq \xi \leq \sigma$. If f is given by (1), then from (2) and (3) we see that

$$I^n_{\xi,\sigma} f(z) = z + \sum_{t=2}^{\infty} \left(\frac{\sigma t + \xi}{\xi + \sigma}\right)^n a_t z^t + (-1)^n \sum_{t=1}^{\infty} \left(\frac{\sigma t - \xi}{\xi + \sigma}\right)^n \overline{b_t z^t}. \tag{4}$$

Also f is given by (1), we get

$$I^n_{\xi,\sigma}f(z) = f * \underbrace{\left(\phi_1(z) + \overline{\phi_2(z)}\right) * ..... * \left(\phi_1(z) + \overline{\phi_2(z)}\right)}_{n\ times}$$
$$= h * \underbrace{(\phi_1(z) * ..... * \phi_2(z))}_{n\ times} + \overline{g * \underbrace{(\phi_1(z) * ..... * \phi_2(z))}_{n\ times}},$$

here "*" denotes power series convolution or Hadamard product and

$$\phi_1(z) = \frac{(\xi+\sigma)z - \xi z^2}{(\xi+\sigma)(1-z)^2}, \qquad \phi_2(z) = \frac{(\xi-\sigma)z - \xi z^2}{(\xi+\sigma)(1-z)^2}.$$

We obtain the following operators investigated by several researchers by specialization of parameters for all $f \in \mathcal{A}$:

(i) $I^n_{0,1}f(z) = D^n f(z)$ ([14]);

(ii) $I^n_\lambda f(z)$ ([6],[5],[10]);

(iii) $I^n_{1,1}f(z) = I^n f(z)$ ([19]) for $f \in H$;

(iv) $I^n_{0,1}f(z) = D^n f(z)$ ([12]);

(v) $I^n_{\xi,1}f(z) = I^n_\xi f(z)$ ([21]).

Now, here we introduced the class $\mathcal{S}_\mathcal{H}(\xi,\sigma,n,\rho,\delta)$ containing the function f defined by (1) satisfying the the condition

$$\mathcal{R}e\left\{\left(1+\rho e^{i\eta}\right)\frac{I^{n+1}_{\xi,\sigma}f(z)}{I^n_{\xi,\sigma}f(z)} - \rho e^{i\eta}\right\} \geq \delta \tag{5}$$

where,$0 \leq \delta < 1, \eta \in \mathbb{R}, \rho \geq 0, n, \lambda \in N_0$ and $I^n_{\xi,\sigma}f(z)$ is defined by (4).

We let the subclass $\overline{S_H}(\xi,\sigma,n,\rho,\delta)$ containing the harmonic functions $f_n(z) = h(z) + \overline{g_n(z)}$ with h and g_n are

$$h(z) = z - \sum_{t=2}^{\infty} a_t z^t \qquad and \qquad g_n(z) = (-1)^n \sum_{t=1}^{\infty} b_t z^t, \qquad a_t, b_t \geq 0. \tag{6}$$

By using the suitable values of the parameters, the classes $\mathcal{S}_\mathcal{H}(\xi,\sigma,n,\rho,\delta)$ reduces to different subclasses of harmonic univalent functions. Such as,

(i) $\mathcal{S}_\mathcal{H}(0,1,0,0,0) = {\mathcal{S}_\mathcal{H}}^*(0)$ in ([2], [17], [18]);

(ii) $\mathcal{S}_\mathcal{H}(0,1,0,0,\delta) = {\mathcal{S}_\mathcal{H}}^*(\delta)$ in ([11]);

(iii) $\mathcal{S}_\mathcal{H}(0,1,1,0,0) = K_H(0)$ in ([2], [17], [18]);

(iv) $\mathcal{S}_\mathcal{H}(0,1,1,0,\delta) = K_H(\delta)$ in ([11]);

(v) $\mathcal{S}_\mathcal{H}(0,1,n,0,\delta) = H(n,\delta)$ in ([12]);

(vi) $\mathcal{S}_{\mathcal{H}}(\xi, 1, n, 0, \delta) = \mathcal{S}_{\mathcal{H}}(\xi, n, \delta)$ in ([21]);

(vii) $\mathcal{S}_{\mathcal{H}}(\xi, \sigma, n, 0, \delta) = \mathcal{S}_{\mathcal{H}}(\xi, \sigma, n, \delta)$ in ([3]).

Define ${S_H}^0(\xi, \sigma, n, \rho, \delta) = S_H(\xi, \sigma, n, \rho, \delta) \cap {S_H}^0$ and $\overline{S_H}^0(\xi, \sigma, n, \rho, \delta) = \overline{S_H}(\xi, \sigma, n, \rho, \delta) \cap {S_H}^0$.

2 Coefficient Condition

2.1 Theorem

If $f = h + \overline{g}$ such that h and g are given by (1) with $b_1 = 0$. Moreover, let

$$\begin{aligned} &\sum_{t=2}^{\infty} \left[\frac{(\sigma t + \xi) + \rho\sigma(t-1)}{\xi + \sigma} - \delta \right] \left[\frac{\sigma t + \xi}{\xi + \sigma} \right]^n |a_t| + \\ &\sum_{t=2}^{\infty} \left[\frac{(\sigma t - \xi) + \rho\sigma(t+1)}{\xi + \sigma} + \delta \right] \left[\frac{\sigma t - \xi}{\xi + \sigma} \right]^n |b_t| \leq (1 - \delta) \end{aligned} \tag{7}$$

where, $0 \leq \xi \leq \sigma/2, n \in \mathcal{N}_0, \rho \geq 0, \frac{\xi}{\xi+\sigma} \leq \delta \leq \frac{\sigma}{\xi+\sigma}$. Then f is sense-preserving, harmonic univalent in disc $\mathcal{U}$ and $f \in {\mathcal{S}_{\mathcal{H}}}^0(\xi, \sigma, n, \rho, \delta)$.

Proof. If z_1 and z_2 are two distinct points then,

$$\begin{aligned} \left| \frac{f(z_1) - f(z_2)}{h(z_1) - h(z_2)} \right| &\geq 1 - \left| \frac{g(z_1) - g(z_2)}{h(z_1) - h(z_2)} \right| \\ &= 1 - \left| \frac{\sum_{t=1}^{\infty} b_t \left(z_1^t - z_2^t\right)}{(z_1 - z_2) + \sum_{t=2}^{\infty} a_t \left(z_1^t - z_2^t\right)} \right| \\ &> 1 - \frac{\sum_{t=1}^{\infty} t\,|b_t|}{1 - \sum_{t=2}^{\infty} t\,|a_t|} \\ &\geq 1 - \frac{\sum_{t=1}^{\infty} \frac{\left[\frac{(\sigma t - \xi) + \rho\sigma(t+1)}{\xi+\sigma} + \delta\right]\left[\frac{\sigma t - \xi}{\xi+\sigma}\right]^n}{1-\delta} |b_t|}{1 - \sum_{t=2}^{\infty} \frac{\left[\frac{(\sigma t + \xi) + \rho\sigma(t-1)}{\xi+\sigma} - \delta\right]\left[\frac{\sigma t + \xi}{\xi+\sigma}\right]^n}{1-\delta} |a_t|} \\ &\geq 0 \end{aligned}$$

this shows univalence. Note that f is sense preserving in init disc $\mathbb{U}$ because,

$$\begin{aligned} \left| h^{'}(z) \right| &\geq 1 - \sum_{t=2}^{\infty} t\,|a_t|\,|z|^{t-1} \\ &> 1 - \sum_{t=2}^{\infty} \frac{1}{1-\delta} \left[\frac{(\sigma t + \xi) + \rho\sigma(t-1)}{\xi + \sigma} - \delta \right] \left[\frac{\sigma t + \xi}{\xi + \sigma} \right]^n |a_t|. \\ &\geq \sum_{t=2}^{\infty} \frac{1}{1-\delta} \left[\frac{(\sigma t - \xi) + \rho\sigma(t+1)}{\xi + \sigma} + \delta \right] \left[\frac{\sigma t - \xi}{\xi + \sigma} \right]^n |b_t| \\ &> \sum_{t=2}^{\infty} t\,|b_t|\,|z|^{t-1} \\ &\geq \left| g^{'}(z) \right|. \end{aligned}$$

Utilizing the obvious truth that $|1-\sigma+w| \geq |1+\sigma-w|$ if and only if $\mathcal{R}(\omega) \geq \delta$, it is sufficient to prove that $|1-\delta+\omega| - |1+\delta-\omega| \geq 0$ gives,

$$\begin{aligned} &\left|(1-\delta-\rho e^{i\eta})I^n_{\xi,\sigma}f(z) + (1+\rho e^{i\eta})I^{n+1}_{\xi,\sigma}f(z)\right| - \\ &\left|(1+\delta+\rho e^{i\eta})I^n_{\xi,\sigma}f(z) - (1+\rho e^{i\eta})I^{n+1}_{\xi,\sigma}f(z)\right| \geq 0. \end{aligned} \tag{8}$$

$$\begin{aligned} =&\left| \begin{matrix} (1-\delta-\rho e^{i\eta})\left(z+\sum_{t=2}^{\infty}\left(\frac{\sigma t+\xi}{\xi+\sigma}\right)^n a_t z^t + (-1)^n \sum_{t=1}^{\infty}\left(\frac{\sigma t-\xi}{\xi+\sigma}\right)^n \overline{b_t z^t}\right) \\ +(1+\rho e^{i\eta})\left(z+\sum_{t=2}^{\infty}\left(\frac{\sigma t+\xi}{\xi+\sigma}\right)^{n+1} a_t z^t + (-1)^{n+1} \sum_{t=1}^{\infty}\left(\frac{\sigma t-\xi}{\xi+\sigma}\right)^{n+1} \overline{b_t z^t}\right) \end{matrix} \right| - \\ &\left| \begin{matrix} (1+\delta+\rho e^{i\eta})\left(z+\sum_{t=2}^{\infty}\left(\frac{\sigma t+\xi}{\xi+\sigma}\right)^n a_t z^t + (-1)^n \sum_{t=1}^{\infty}\left(\frac{\sigma t-\xi}{\xi+\sigma}\right)^n \overline{b_t z^t}\right) \\ -(1+\rho e^{i\eta})\left(z+\sum_{t=2}^{\infty}\left(\frac{\sigma t+\xi}{\xi+\sigma}\right)^{n+1} a_t z^t + (-1)^{n+1} \sum_{t=1}^{\infty}\left(\frac{\sigma t-\xi}{\xi+\sigma}\right)^{n+1} \overline{b_t z^t}\right) \end{matrix} \right| \geq 0 \\ &\geq 2(1-\delta)|z| - \sum_{t=2}^{\infty}\left[\frac{(\sigma t+\xi)+\rho\sigma(t-1)}{\xi+\sigma}+1-\delta\right]\left[\frac{\sigma t+\xi}{\xi+\sigma}\right]^n |a_t||z|^t \\ &\quad - \sum_{t=2}^{\infty}\left[\frac{(\sigma t-\xi)+\rho\sigma(t+1)}{\xi+\sigma}-1+\delta\right]\left[\frac{\sigma t-\xi}{\xi+\sigma}\right]^n |b_t||z|^t \\ &\quad - \sum_{t=2}^{\infty}\left[\frac{(\sigma t+\xi)+\rho\sigma(t-1)}{\xi+\sigma}-1-\delta\right]\left[\frac{\sigma t+\xi}{\xi+\sigma}\right]^n |a_t||z|^t \\ &\quad - \sum_{t=2}^{\infty}\left[\frac{(\sigma t-\xi)+\rho\sigma(t+1)}{\xi+\sigma}+1+\delta\right]\left[\frac{\sigma t-\xi}{\xi+\sigma}\right]^n |b_t||z|^t \\ &> 2(1-\delta)|z| \left\{ \begin{matrix} 1-\sum_{t=2}^{\infty}\frac{1}{1-\delta}\left[\frac{(\sigma t+\xi)+\rho\sigma(t-1)}{\xi+\sigma}-\delta\right]\left[\frac{\sigma t+\xi}{\xi+\sigma}\right]^n |a_t| \\ -\sum_{t=2}^{\infty}\frac{1}{1-\delta}\left[\frac{(\sigma t-\xi)+\rho\sigma(t+1)}{\xi+\sigma}+\delta\right]\left[\frac{\sigma t-\xi}{\xi+\sigma}\right]^n |b_t| \end{matrix} \right\}. \end{aligned}$$

The last inequality is non-negative by (7), and this completes the proof.

2.2 Theorem

If $f_n = h + \overline{g_n}$ defined by (6) with $b_1 = 0$. Then $f_n \in \overline{\mathcal{S}_{\mathcal{H}}}^0(\xi,\sigma,n,\rho,\delta)$ if and only if

$$\begin{aligned} &\sum_{t=2}^{\infty}\left[\frac{(\sigma t+\xi)+\rho\sigma(t-1)}{\xi+\sigma}-\delta\right]\left[\frac{\sigma t+\xi}{\xi+\sigma}\right]^n a_t + \\ &\sum_{t=2}^{\infty}\left[\frac{(\sigma t-\xi)+\rho\sigma(t+1)}{\xi+\sigma}+\delta\right]\left[\frac{\sigma t-\xi}{\xi+\sigma}\right]^n b_t \leq (1-\delta) \end{aligned} \tag{9}$$

where, $0 \leq \xi \leq \sigma/2, n \in \mathcal{N}_0, \rho \geq 0, \frac{\xi}{\xi+\sigma} \leq \delta \leq \frac{\sigma}{\xi+\sigma}$.

Proof. We skipped the "if" part because it is similar to Theorem 2.1, and we can write

$$\overline{\mathcal{S}_{\mathcal{H}}}^{0}(\xi,\sigma,n,\rho,\delta) \subset \mathcal{S}_{\mathcal{H}}{}^{0}(\xi,\sigma,n,\rho,\delta).$$

For the "only if" part, we prove that $f_n \notin \overline{\mathcal{S}_{\mathcal{H}}}^{0}(\xi,\sigma,n,\rho,\delta)$ if the condition (9) does not hold. Note that a necessary and sufficient condition for $f_n(z) = h(z) + \overline{g_n(z)}$ defined by (6), belongs to $\overline{\mathcal{S}_{\mathcal{H}}}^{0}(\xi,\sigma,n,\rho,\delta)$ and satisfies the condition (5). This is analogous to

$$\mathcal{R}e\left\{\frac{(1-\delta)z-\left(\begin{array}{c}\sum_{t=2}^{\infty}\left[\frac{(\sigma t+\xi)+\rho\sigma(t-1)}{\xi+\sigma}-\delta\right]\left[\frac{\sigma t+\xi}{\xi+\sigma}\right]^{n}a_t z^t+\\ \sum_{t=2}^{\infty}\left[\frac{(\sigma t-\xi)+\rho\sigma(t+1)}{\xi+\sigma}+\delta\right]\left[\frac{\sigma t-\xi}{\xi+\sigma}\right]^{n}b_t\overline{z}^t\end{array}\right)}{z-\sum_{t=2}^{\infty}\left[\frac{\sigma t+\xi}{\xi+\sigma}\right]^{n}a_t z^t+\sum_{t=2}^{\infty}\left[\frac{\sigma t-\xi}{\xi+\sigma}\right]^{n}b_t\overline{z}^t}\right\}\geq 0.$$

For every value of z, $|z| = r < 1$ must hold the above condition. After selecting the values on the positive real axis for z. where $0 \leq z = r < 1$, we must take

$$\mathcal{R}e\left\{\frac{(1-\delta)-\left(\begin{array}{c}\sum_{t=2}^{\infty}\left[\frac{(\sigma t+\xi)+\rho\sigma(t-1)}{\xi+\sigma}-\delta\right]\left[\frac{\sigma t+\xi}{\xi+\sigma}\right]^{n}a_t r^{t-1}+\\ \sum_{t=2}^{\infty}\left[\frac{(\sigma t-\xi)+\rho\sigma(t+1)}{\xi+\sigma}+\delta\right]\left[\frac{\sigma t-\xi}{\xi+\sigma}\right]^{n}b_t r^{t-1}\end{array}\right)}{1-\sum_{t=2}^{\infty}\left[\frac{\sigma t+\xi}{\xi+\sigma}\right]^{n}a_t r^{t-1}+\sum_{t=2}^{\infty}\left[\frac{\sigma t-\xi}{\xi+\sigma}\right]^{n}b_t b_t r^{t-1}}\right\}\geq 0. \tag{10}$$

If the above condition (9) does not hold, then numerator of (10) is negative for r sufficiently close to 1. Hence there exist $z_0 = r_0$ in $(0, 1)$ for which the quotient in (10) is negative. This is contradiction to the required condition of $f_n \in \overline{\mathcal{S}_{\mathcal{H}}}^{0}(\xi,\sigma,n,\rho,\delta)$ and hence here complete the proof.

3 Extreme Points

For investigating the extreme points for function $f_n \in \overline{\mathcal{S}_{\mathcal{H}}}^{0}(\xi,\sigma,n,\rho,\delta)$, we use the coefficient condition obtained in Section 2.

3.1 Theorem

Let f_n be given by (2) then $f_n \in \overline{\mathcal{S}_{\mathcal{H}}}^0(\xi,\sigma,n,\rho,\delta)$ if and only if

$$
\begin{aligned}
f_n(z) &= \sum_{t=1}^{\infty}\left(X_t h_t(z) + Y_t g_{n_t}(z)\right),\\
\mathit{where},\quad h_1(z) &= z,\quad h_t(z) = z - \frac{1-\delta}{\left[\frac{(\sigma t+\xi)+\rho\sigma(t-1)}{\xi+\sigma}-\delta\right]\left[\frac{\sigma t+\xi}{\xi+\sigma}\right]^n} z^t,\\
\mathit{and}\quad g_{n_1}(z) &= z,\quad g_{n_t}(z) = z + (-1)^n \frac{1-\delta}{\left[\frac{(\sigma t-\xi)+\rho\sigma(t-1)}{\xi+\sigma}-\delta\right]\left[\frac{\sigma t-\xi}{\xi+\sigma}\right]^n} \overline{z}^t,
\end{aligned}
$$

$X_t \geq 0, Y_t \geq 0, \sum_{t=1}^{\infty}(X_t+Y_t) = 1, 0 \leq \xi \leq \sigma/2, n \in N_0, \rho \geq 0, \frac{\xi}{\xi+\sigma} \leq \delta \leq \frac{\sigma}{\xi+\sigma}, (t = 2,3,\ldots)$.

Particularly, the extreme points of $f_n \in \overline{\mathcal{S}_{\mathcal{H}}}^0(\xi,\sigma,n,\rho,\delta)$ are $\{h_t\}$ and $\{g_{n_t}\}$.

Proof. We have from equation (6), for functions f_n as,

$$
\begin{aligned}
f_n(z) &= \sum_{t=1}^{\infty}\left(X_t h_t(z) + Y_t g_{n_t}(z)\right)\\
&= \sum_{t=1}^{\infty}(X_t+Y_t)z - \sum_{t=2}^{\infty} \frac{1-\delta}{\left[\frac{(\sigma t+\xi)+\rho\sigma(t-1)}{\xi+\sigma}-\delta\right]\left[\frac{\sigma t+\xi}{\xi+\sigma}\right]^n} X_t z^t\\
&+ (-1)^n \sum_{t=2}^{\infty} \frac{1-\delta}{\left[\frac{(\sigma t-\xi)+\rho\sigma(t-1)}{\xi+\sigma}-\delta\right]\left[\frac{\sigma t-\xi}{\xi+\sigma}\right]^n} Y_t \overline{z}^t.
\end{aligned}
$$

Then

$$
\begin{aligned}
&\sum_{t=2}^{\infty}\left(\frac{\left[\frac{(\sigma t+\xi)+\rho\sigma(t-1)}{\xi+\sigma}-\delta\right]\left[\frac{\sigma t+\xi}{\xi+\sigma}\right]^n}{1-\delta}\right)\left(\frac{1-\delta}{\left[\frac{(\sigma t+\xi)+\rho\sigma(t-1)}{\xi+\sigma}-\delta\right]\left[\frac{\sigma t+\xi}{\xi+\sigma}\right]^n} X_t\right)\\
&+\sum_{t=2}^{\infty}\left(\frac{\left[\frac{(\sigma t-\xi)+\rho\sigma(t+1)}{\xi+\sigma}+\delta\right]\left[\frac{\sigma t-\xi}{\xi+\sigma}\right]^n}{1-\delta}\right)\left(\frac{1-\delta}{\left[\frac{(\sigma t-\xi)+\rho\sigma(t+1)}{\xi+\sigma}+\delta\right]\left[\frac{\sigma t-\xi}{\xi+\sigma}\right]^n} Y_t\right)\\
&= \sum_{t=2}^{\infty} X_t + \sum_{t=2}^{\infty} Y_t = 1 - X_1 - Y_1 \leq 1
\end{aligned}
$$

and so $f_n \in \overline{S_H}^0(\xi,\sigma,n,\rho,\delta)$. Conversely, if $f_n \in \overline{S_H}^0(\xi,\sigma,n,\rho,\delta)$, then

$$a_t \leq \frac{1-\delta}{\left[\frac{(\sigma t+\xi)+\rho\sigma(t-1)}{\xi+\sigma}-\delta\right]\left[\frac{\sigma t+\xi}{\xi+\sigma}\right]^n} \qquad and \qquad b_t \leq \frac{1-\delta}{\left[\frac{(\sigma t-\xi)+\rho\sigma(t+1)}{\xi+\sigma}+\delta\right]\left[\frac{\sigma t-\xi}{\xi+\sigma}\right]^n}.$$

Set

$$X_t = \frac{\left[\frac{(\sigma t+\xi)+\rho\sigma(t-1)}{\xi+\sigma}-\delta\right]\left[\frac{\sigma t+\xi}{\xi+\sigma}\right]^n}{1-\delta}a_t, \quad (t=2,3,\ldots),$$

$$Y_t = \frac{\left[\frac{(\sigma t-\xi)+\rho\sigma(t+1)}{\xi+\sigma}+\delta\right]\left[\frac{\sigma t-\xi}{\xi+\sigma}\right]^n}{1-\delta}b_t, \quad (t=2,3,\ldots),$$

and

$$X_1+Y_1 = 1-\left(\sum_{t=2}^{\infty} X_t+Y_t\right),$$

where $X_t, Y_t \geq 0$. Then, as necessary, we obtain

$$\begin{aligned} f_n(z) &= (X_1+Y_1)z+\sum_{t=2}^{\infty} X_t h_t(z)+\sum_{t=2}^{\infty} Y_t g_{n_t}(z) \\ &= \sum_{t=1}^{\infty}\left(X_t h_t(z)+Y_t g_{n_t}(z)\right). \end{aligned}$$

4 Distortion and Convex Combination

In below theorem we prove that the $\overline{\mathcal{S}_{\mathcal{H}}}^0(\xi,\sigma,n,\rho,\delta)$ remains unchanged under distortion and convex combinations of its numbers.

4.1 Theorem

Let $f_n \in \overline{S_H}^0(\xi,\sigma,n,\rho,\delta)$. Then for $|z|=r<1$ and $0\leq\xi\leq\sigma/2, n\in N_0, \rho\geq 0, \frac{\xi}{\xi+\sigma}\leq\delta\leq\frac{\sigma}{\xi+\sigma}$ we have

$$|f_n(z)| \leq r+\frac{1-\delta}{\left[\frac{(2\sigma+\xi)+2\rho\sigma(t-1)}{\xi+\sigma}-\delta\right]\left[\frac{2\sigma+\xi}{\xi+\sigma}\right]^n}r^2$$

and

$$|f_n(z)| \geq r-\frac{1-\delta}{\left[\frac{(2\sigma+\xi)+2\rho\sigma(t-1)}{\xi+\sigma}-\delta\right]\left[\frac{2\sigma+\xi}{\xi+\sigma}\right]^n}r^2.$$

Proof. The proof of left hand inequality skipped due to similarity. hence we only prove inequality on right hand side. Let $f_n \in \overline{S_H}^0(\xi,\sigma,n,\rho,\delta)$. Taking the absolute value of f_n, we have

$$\begin{aligned}
|f_n(z)| &\le r + \sum_{t=2}^{\infty}(a_t+b_t)r^2 \\
&\le r + \frac{(1-\delta)r^2}{\left[\frac{(2\sigma+\xi)+2\rho\sigma(t-1)}{\xi+\sigma}-\delta\right]\left[\frac{2\sigma+\xi}{\xi+\sigma}\right]^n}\sum_{t=2}^{\infty}\left\{\begin{array}{c}\frac{\left[\frac{(\sigma t+\xi)+\rho\sigma(t-1)}{\xi+\sigma}-\delta\right]\left[\frac{\sigma t+\xi}{\xi+\sigma}\right]^n}{1-\delta}a_t+ \\ \frac{\left[\frac{(\sigma t-\xi)+\rho\sigma(t+1)}{\xi+\sigma}+\delta\right]\left[\frac{\sigma t-\xi}{\xi+\sigma}\right]^n}{1-\delta}b_t\end{array}\right\} \\
&\le r + \frac{(1-\delta)}{\left[\frac{(2\sigma+\xi)+2\rho\sigma(t-1)}{\xi+\sigma}-\delta\right]\left[\frac{2\sigma+\xi}{\xi+\sigma}\right]^n}r^2.
\end{aligned}$$

The left hand inequality in Theorem 4.1 covers the following result.

4.2 Corollary

Let f_n of type (6) be so that, $f_n \in \overline{S_H}^0(\xi,\sigma,n,\rho,\delta)$, where $0\le\xi\le\sigma/2,\ n\in N_0,\ \rho\ge 0$, $\frac{\xi}{\xi+\sigma}\le\delta\le\frac{\sigma}{\xi+\sigma}$. Then

$$\left\{\omega : |\omega| < 1 - \frac{1-\delta}{\left[\frac{(2\sigma+\xi)+2\rho\sigma(t-1)}{\xi+\sigma}-\delta\right]\left[\frac{2\sigma+\xi}{\xi+\sigma}\right]^n}\right\} \subset f_n(U).$$

4.3 Theorem

The class $f_n \in \overline{S_H}^0(\xi,\sigma,n,\rho,\delta)$ is closed under convex combinations.

Proof. Let $f_{n_j} \in \overline{S_H}^0(\xi,\sigma,n,\rho,\delta)$ for j=1,2,..., where f_{n_j} is given by

$$f_{n_j}(z) = z - \sum_{t=2}^{\infty}a_{t_j}z^t + (-1)^n\sum_{t=2}^{\infty}b_{t_j}\overline{z}^t.$$

Then by (9),

$$\sum_{t=2}^{\infty}\frac{\left[\frac{(\sigma t+\xi)+\rho\sigma(t-1)}{\xi+\sigma}-\delta\right]\left[\frac{\sigma t+\xi}{\xi+\sigma}\right]^n}{1-\delta}a_{t_j} + \sum_{t=2}^{\infty}\frac{\left[\frac{(\sigma t-\xi)+\rho\sigma(t+1)}{\xi+\sigma}+\delta\right]\left[\frac{\sigma t-\xi}{\xi+\sigma}\right]^n}{1-\delta}b_{t_j} \le 1. \tag{11}$$

For $\sum_{j=1}^{\infty}p_j = 1, 0<p_j<1$, we write the convex combination of f_{n_j} as

$$\sum_{j=1}^{\infty}p_j f_{n_j}(z) = z - \sum_{t=2}^{\infty}\left(\sum_{j=1}^{\infty}p_j a_{t_j}\right)z^t + (-1)^n\sum_{t=2}^{\infty}\left(\sum_{j=1}^{\infty}p_j b_{t_j}\right)\overline{z}^t.$$

Then by (11),

$$\sum_{t=2}^{\infty} \frac{\left[\frac{(\sigma t+\xi)+\rho\sigma(t-1)}{\xi+\sigma} - \delta\right]\left[\frac{\sigma t+\xi}{\xi+\sigma}\right]^n}{1-\delta}\left(\sum_{j=1}^{\infty} p_j a_{t_j}\right) + \sum_{t=2}^{\infty} \frac{\left[\frac{(\sigma t-\xi)+\rho\sigma(t+1)}{\xi+\sigma} + \delta\right]\left[\frac{\sigma t-\xi}{\xi+\sigma}\right]^n}{1-\delta}\left(\sum_{j=1}^{\infty} p_j b_{t_j}\right)$$
$$= \sum_{j=1}^{\infty} p_j \left\{\sum_{t=2}^{\infty} \frac{\left[\frac{(\sigma t+\xi)+\rho\sigma(t-1)}{\xi+\sigma} - \delta\right]\left[\frac{\sigma t+\xi}{\xi+\sigma}\right]^n}{1-\delta} a_{t_j} + \sum_{t=2}^{\infty} \frac{\left[\frac{(\sigma t-\xi)+\rho\sigma(t+1)}{\xi+\sigma} + \delta\right]\left[\frac{\sigma t-\xi}{\xi+\sigma}\right]^n}{1-\delta} b_{t_j}\right\}$$
$$\leq \sum_{j=1}^{\infty} p_j = 1.$$

This is the condition required by (9) and so $\sum_{j=1}^{\infty} p_j f_{n_j}(z) \in \overline{S_H}^0(\xi, \sigma, n, \rho, \delta)$.

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References

[1] O. P. Ahuja (2005). Planar harmonic univalent and related mappings. *Journal of Inequalities in Pure and Applied Mathematics*, *6*(4), 1–18.

[2] Y. Avci & E. Zlotkiewicz (1991). On harmonic univalent mappings. *Annales Universitatis Mariae Curie-Sklodowska, sectio A - Mathematica*, *44*, 1–7.

[3] H. Bayram & S. Yalçın (2017). A subclass of harmonic univalent functions defined by a linear operator. *Palestine Journal of Mathematics*, *6*(Special Issue: II), 320–326.

[4] H. Bayram & S. Yalçın (2020). On a new subclass of harmonic univalent functions. *Malaysian Journal of Mathematical Sciences*, *14*(1), 63–75.

[5] N. E. Cho & H. M. Srivastava (2003). Argument estimates of certain analytic functions defined by a class of multiplier transformations. *Mathematical and Computer Modelling*, *37*(1-2), 39–49. https://doi.org/10.1016/S0895-7177(03)80004-3.

[6] N.-E. Cho & T.-H. Kim (2003). Multiplier transformations and strongly close-to-convex functions. *Bulletin of the Korean Mathematical Society*, *40*(3), 399–410. https://doi.org/10.4134/BKMS.2003.40.3.399.

[7] J. Clunie & T. Sheil-Small (1984). Harmonic univalent functions. *Annales Academiae Scientiarum Fennicae Mathematica*, *9*, 3–25.

[8] M. Darus & K. Al-Shaqsi (2006). On harmonic univalent functions defined by a generalized Ruscheweyh derivatives operator. *Lobachevskii Journal of Mathematics*, *22*, 19–26.

[9] P. Duren (2004). *Harmonic mappings in the plane*. Cambridge University Press, Cambridge, United Kingdom.

[10] T. M. Flett (1972). The dual of an inequality of Hardy and Littlewood and some related inequalities. *Journal of Mathematical Analysis and Applications*, *38*(3), 746–765.

[11] J. M. Jahangiri, G. Murugusundaramoorthy & K. Vijaya (2002). Salagean-type harmonic univalent functions. *South J. Pure Appl. Math.*, *2*(6), 77–82.

[12] J. M. Jahangiri (1999). Harmonic functions starlike in the unit disk. *Journal of Mathematical Analysis and Applications*, *235*(2), 470–477. https://doi.org/10.1006/jmaa.1999.6377.

[13] A. N. Metkari, N. D. Sangle & S. P. Hande (2020). On the generalization of a class of harmonic univalent functions defined by a linear operator. *South Asian Journal of Mathematics*, *10*(1), 21–32.

[14] G. S. Salagean (1983). Subclasses of univalent functions. In *Complex Analysis – Fifth Romanian-Finnish Seminar. Lecture Notes in Mathematics*, pp. 362–372. Springer, Berlin, Heidelberg.

[15] K. A. Shaqsi & M. Darus (2007). On subclass of harmonic starlike functions with respect to k -symmetric points. *International Mathematical Forum*, *2*(57), 2799–2805.

[16] K. A. Shaqsi & M. Darus (2008). On harmonic univalent functions with respect to k -symmetric points. *International Journal of Contemporary Mathematical Sciences*, *3*(3), 111–118.

[17] H. Silverman & E. M. Silvia (1999). Subclasses of harmonic univalent functions. *New Zealand Journal of Mathematics*, *28*, 275–284.

[18] H. Silverman (1998). Harmonic univalent functions with negative coefficients. *Journal of Mathematical Analysis and Applications*, *220*(1), 283–289. https://doi.org/10.1006/jmaa.1997.5882.

[19] B. A. Uralegaddi & C. Somanatha (1992). Certain classes of univalent functions. *Current Topics in Analytic Function Theory*, pp. 371–374. https://doi.org/10.1142/9789814355896_0032.

[20] Y. P. Yadav & N. D. Sangle (2012). A class of harmonic univalent functions with varying arguments defined by generalized derivative operator. *International Journal of Modern Engineering Research*, *2*(4), 1934–1939.

[21] E. Yasar & S. Yalçin (2013). Certain properties of a subclass of harmonic functions. *Applied Mathematics & Information Sciences*, *7*(5), 1749–1753. https://doi.org/10.12785/amis/070512.

[22] A. Yusuf & M. Darus (2019). On subclass of harmonic univalent functions defined by a generalised operator. *International Journal of Mathematics and Computer Science*, *14*(2), 465–472.